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Some Remarks on the Degree of Monotone Approximation

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1. INTRODUCTION

Many articles have been written on the degree of convergence of monotone approximation. See Shisha [7], Roulier [5] and [6], Lorentz and Zeller [1] and [2], and Lorentz [3].

The problem is as follows. Let $1 \leqslant k_1 < k_2 < \cdots < k_p$ be integers and $\epsilon_1, \ldots, \epsilon_p$ corresponding signs (± 1). For each nonnegative integer n let H_n be the set of all polynomials of degree less than or equal to n. Let M_n be the set of all those polynomials p in H_n which satisfy $\epsilon_i p^{(k_i)}(x) \geqslant 0$ for $a \leqslant x \leqslant b$ and $i = 1, \ldots, p$.

If $f \in C[a, b]$ define $D_n(f) = \inf_{p \in M_n} \| f - p \|$, where $\| \cdot \|$ is the uniform norm on $[a, b]$. Let $E_n(f) = \inf_{p \in H_n} \| f - p \|$. If $\epsilon_i f^{(k_i)}(x) \geqslant 0$ for $a \leqslant x \leqslant b$ and $i = 1, \ldots, p$ then we seek upper bounds for $D_n(f)$. Most of the estimates obtained to date have been restricted to $p = 1$, $k_1 = k$, $\epsilon_1 = 1$ and are not best possible. Lorentz and Zeller in [2] show that there is an $f \in C[a, b]$ with $f^{(k)}(x) \geqslant 0$ on $[a, b]$ for which

$$\overline{\lim_{n \to \infty}} D_n(f)/E_n(f) = +\infty.$$

On the other hand Lorentz in [3] conjectures that for function f satisfying $\epsilon_i f^{(k_i)}(x) > 0$ for $a \leqslant x \leqslant b$ and $i = 1, \ldots, p$ we have

$$D_n(f)/E_n(f) \text{ bounded.}$$

Roulier in [5] and [6] studies these cases. In [6] Roulier finds sufficient conditions on f to insure that for n sufficiently large

$$D_n(f) = E_n(f).$$

In [5] Roulier obtains sufficient conditions to insure that $D_n(f)/E_n(f)$ is bounded but the bound obtained depends on the range of f'. It is the purpose

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of this paper to use the result of [6] to make an observation on "best possible" estimates in the Jackson sense and to improve the results from [5] obtaining a bound on $D_n(f)/E_n(f)$ independent of the range of f'.

2. The First Result

In this section we consider the case $p = 1$, $k_1 = 1$, $\epsilon_1 = 1$. We first give two lemmas.

Lemma 1. *Let $f' \in C[a, b]$ and assume that $0 \leqslant f'(x) \leqslant M$ on $[a, b]$. Then, given any x in $[a, b]$, there exist constants η and ξ in (a, b) so that*

$$\int_a^b (f(x) - f(t))\, dt = \tfrac{1}{2}[f'(\eta)(x-a)^2 - f'(\xi)(x-b)^2].$$

Moreover,

$$\left| \int_a^b (f(x) - f(t))\, dt \right| \leqslant \tfrac{1}{2}M(b-a)^2.$$

Proof. Observe that

$$\begin{aligned}\int_a^b (f(x) - f(t))\, dt &= \int_a^b f[x, t](x - t)\, dt \\ &= \int_{x-b}^{x-a} f[x, x-u]u\, du \\ &= \int_0^{x-a} f[x, x-u]u\, du + \int_{x-b}^0 f[x, x-u]u\, du.\end{aligned}$$

Lemma 2. *Let $f' \in C[0, 1]$ and assume that $0 < d \leqslant f'(x) \leqslant M$ on $[0, 1]$. Then for n sufficiently large we have*

$$D_n(f) \leqslant ((M/d) + 1)\, E_n(f).$$

Proof. The proof is the same as that of Theorem 1 in [5] using Lemma 1 and the fact that if S_n is the polynomial of best approximation to f on $[0, 1]$ then $S_n(x_2) - S_n(x_1) \geqslant 0$ if $x_2 - x_1 \geqslant 2E_n(f)/d$.

Theorem 1. *Let $f' \in C[0, 1]$ and assume that $f'(x) \geqslant d > 0$ on $[0, 1]$. Then, if f is not a polynomial,*

$$\overline{\lim_{n\to\infty}}\, D_n(f)/E_n(f) \leqslant 2.$$

Proof. Let $\alpha \geqslant 1$ be fixed. Choose m so large that $E_m(f') < d/(3 + \alpha)$. Let p_m be the polynomial from H_m of best approximation to f' on $[0, 1]$. Let $h(x) = f'(x) - p_m(x) + (1 + \alpha) E_m(f')$. Then we have

$$\alpha E_m(f') \leqslant h(x) \leqslant (2 + \alpha) E_m(f'). \tag{1}$$

Now let

$$\phi(x) = \int_0^x h(t)\, dt.$$

Thus we have

$$\phi(x) = f(x) - Q_{m+1}(x), \tag{2}$$

where

$$Q_{m+1}(x) = f(0) + \int_0^x (p_m(t) - (1 + \alpha) E_m(f'))\, dt. \tag{3}$$

We also see that

$$\begin{aligned} Q'_{m+1}(x) &= p_m(x) - (1 + \alpha) E_m(f') \\ &= p_m(x) - f'(x) + f'(x) - (1 + \alpha) E_m(f') \\ &\geqslant f'(x) - (2 + \alpha) E_m(f') \\ &\geqslant d - (2 + \alpha)d/(3 + \alpha) \\ &= d/(3 + \alpha). \end{aligned}$$

From (1) and the fundamental theorem of calculus we have

$$\alpha E_m(f') \leqslant \phi'(x) \leqslant (2 + \alpha) E_m(f').$$

By Lemma 2 we have for n sufficiently large

$$D_n(\phi) \leqslant \left(\frac{2 + \alpha}{\alpha} + 1\right) E_n(\phi).$$

That is, for n sufficiently large,

$$D_n(\phi) \leqslant 2(1 + (1/\alpha)) E_n(\phi).$$

If in addition $n \geqslant m + 1$, we have from (2) and the monotonicity of Q_{m+1}

$$E_n(\phi) = E_n(f) \qquad \text{and} \qquad D_n(\phi) \geqslant D_n(f).$$

This, together with (4), gives

$$D_n(f)/E_n(f) \leqslant 2(1 + (1/\alpha)) \qquad \text{for } n$$

sufficiently large. But α can be chosen as large as desired. This completes the proof.

3. Best Possible Estimates

If f has infinitely many continuous derivatives on $[a, b]$ and if $\epsilon_i f^{(k_i)}(x) > 0$ on $[a, b]$ for $i = 1,..., p$ then it follows as a special case of the main theorem in [6] that for n sufficiently large we have

$$E_n(f) = D_n(f).$$

So we may as well for further study assume that there is a $k \geqslant k_p$ for which $f^{(k)}$ is continuous on $[a, b]$ and $f^{(k+1)}$ is not continuous on $[a, b]$. For simplicity in what follows we will work on the interval $[-1, +1]$.

Definition. For $-1 \leqslant x \leqslant 1$ and $n = 1, 2,...$ define

$$\Delta_n(x) = \max\left(\frac{(1 - x^2)^{1/2}}{n}, \frac{1}{n^2}\right)$$

and $\Delta_0(x) = 1$.

Theorem 2. *Let* $1 \leqslant k_1 < k_2 < \cdots < k_p$ *be* p *fixed integers and* $\epsilon_1, ..., \epsilon_p$ *fixed signs. Assume* $\epsilon_i f^{(k_i)}(x) > 0$ *on* $[-1, +1]$ *for* $i = 1, 2,..., p$. *We also assume that for some integer* $k \geqslant k_p$ *we have* $f^{(k)} \in C[-1, +1]$ *but* $f^{(k+1)} \notin C[-1, +1]$. *With these assumptions we can conclude that there are polynomials* $P_n \in H_n$ *such that*

$$|f(x) - P_n(x)| \leqslant C_k \Delta_n(x)^k w(f^{(k)}, \Delta_n(x)) \tag{5}$$

and for n *sufficiently large we have* $\epsilon_i P_n^{(k_i)}(x) > 0$ *for* $-1 \leqslant x \leqslant 1$ *and* $i = 1, 2,..., p$.

Moreover, this result is best possible in the sense that no sequence of polynomials $P_n \in H_n$ can satisfy (5) if we replace $\Delta_n(x)^k$ in the right side of (5) by $\Delta_n(x)^{k+\epsilon}$ for some $\epsilon > 0$.

The proof is an easy consequence of the theorem in [8] and Theorem 6 in [4, p. 75].

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